Assignment 13.

This homework is due *Thursday*, December 10.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems is due December 5.

1. Quick reminder

A. A *measure space* is a triple $(X, \mathcal{M}, \mu)$, where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and $\mu : \mathcal{M} \to [0, \infty]$ is a function, called *measure*, with the following properties: $\mu(\emptyset) = 0$, $\mu(\cup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$ whenever $\{E_k\}_{k=1}^{\infty}$ is a disjoint collection of sets in $\mathcal{M}$.

For a set X, a function $\mu^* : 2^X \to [0, \infty]$ is called an *outer measure* on X if $\mu^*(\emptyset) = 0$ and μ^* is countable monotone, i.e. $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$ whenever $E \subseteq \bigcup_{k=1}^{\infty} E_k$.

B. Given any outer measure μ^*, we say that a set $E \subseteq X$ is measurable w.r.t. μ^* if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C)$ for every $A \subseteq X$.

One can show that sets measurable w.r.t. μ^* form a σ-algebra $\mathcal{M}$. One can show that the restriction $\bar{\mu}$ of μ^* to $\mathcal{M}$ is a measure.

C. For any $S \subseteq 2^X$, given any set function $\mu : S \to [0, \infty]$, one can define a function $\mu^* : 2^X \to [0, \infty]$ as follows:

$$\mu^*(E) = \inf \left\{ \sum \mu(E_k) \,\middle|\, E_k \in S, E \subseteq \bigcup E_k \right\}.$$

One can prove that such μ^* is an outer measure, called the *outer measure induced by μ*.

D. Thus, given any set function $\mu : S \to [0, \infty]$ on any $S \subseteq 2^X$, one can construct the outer measure μ^* induced by μ (see **C**) and then define measurable sets and the measure $\bar{\mu}$ (see **B**). The latter is called the *Carathéodory measure induced by μ*.

$$\boxed{\mu : S \to [0, \infty]} \dashrightarrow \boxed{\mu^* : 2^X \to [0, \infty]} \dashrightarrow \boxed{\bar{\mu} : \mathcal{M} \to [0, \infty]}$$

Note that without imposing additional requirements on S and μ there is no guarantee that $\bar{\mu}$ extends μ. That is, there is no guarantee that the sets in S will turn out to be measurable, nor, if $E \in S$ happens to be measurable, that $\bar{\mu}(E) = \mu(E)$.

2. Exercises

(1) (Prop. 17.1.1) Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that μ has the following properties.

(a) (Finite Additivity) For any finite disjoint collection $E_1, \ldots, E_n$ of measurable sets,

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

(b) (Monotonicity) If $A \subseteq B$ are measurable sets then $\mu(A) \leq \mu(B)$.

(c) (Excision) If additionally $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(d) (Countable Monotonicity) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets that covers a measurable set E,

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

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(2) (17.1.7i) Prove that if $(X, \mathcal{M})$ is a measurable space and μ, ν are measures on $\mathcal{M}$, then $\mu + \nu$ defined by $(\mu + \nu)(E) = \mu(E) + \nu(E)$ is also a measure on $\mathcal{M}$.

(3) (17.4.19) Let μ^* be an outer measure on X. Show that if $\mu^*(E) = 0$ for some $E \subseteq X$, then E is measurable w.r.t μ^*, every subset $E' \subseteq E$ is measurable w.r.t μ^*, and $\mu^*(E') = 0$.

(4) (17.4.22) On the collection $S = \{\emptyset, [1, 2]\}$ of subsets of $\mathbb{R}$, define the set function $\mu : S \to [0, \infty]$ as follows: $\mu(\emptyset) = 0, \mu([1, 2]) = 1$. Determine the outer measure μ^* induced by μ and the σ-algebra of measurable sets.

(5) (17.4.23) On the collection S of all subsets of $\mathbb{R}$, define the set function $\mu : S \to [0, \infty]$ by setting $\mu(A)$ to be the number of integer points in A. Determine the outer measure μ^* induced by μ and the σ-algebra of measurable sets.

(6) (17.5.26) Consider the collection $S = \{\emptyset, [0, 1], [0, 3], [2, 3]\}$ of subsets of $\mathbb{R}$ and define $\mu(\emptyset) = 0, \mu([0, 1]) = 1, \mu([0, 3]) = 1, \mu([2, 3]) = 1$. Show that $\mu : S \to [0, \infty]$ is a finitely additive. Can μ be extended to a measure? What are the subsets of $\mathbb{R}$ that are measurable with respect to the outer measure μ^* induced by μ?

(7) (17.5.27) Let S be a collection of subsets of a set X and $\mu : S \to [0, \infty]$ a set function. Show that μ is countably monotone if and only if μ^* is an extension of μ (i.e. $\mu^*(A) = \mu(A)$ for every $A \in S$.)